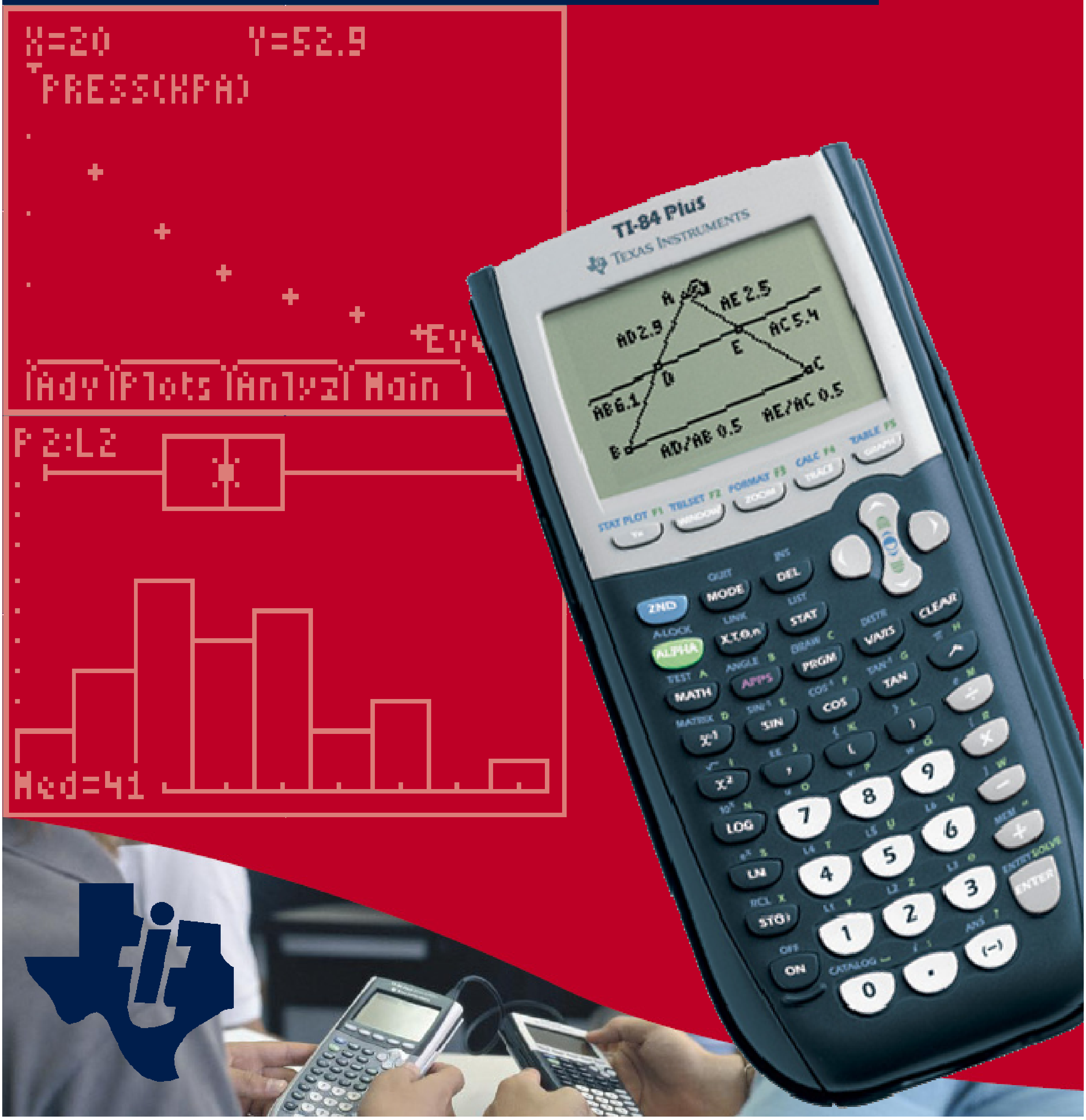
Koen Stulens

Exploring mathematics

with a graphing calculator, an introduction



Exploring mathematics

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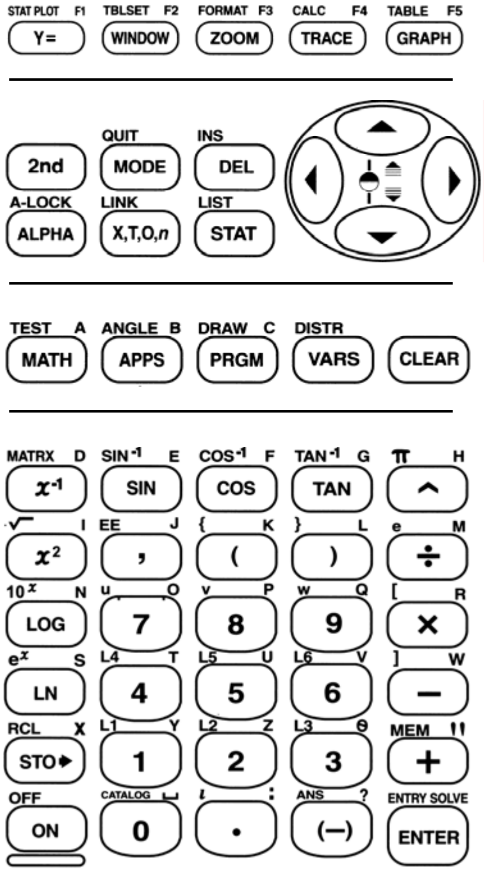
TEXAS INSTRUMENTS

TI Technology – Beyond Numbers

Content

1 The keys



Graphing keys

Edit keys

Advanced Function keys

Scientific keys

A key on the **`TI-83/84 Plus`** has mostly several functions.

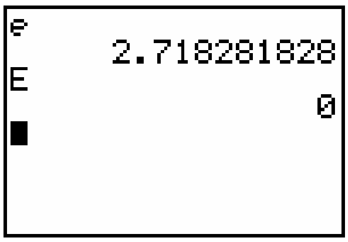
Primary function: on the key

Secondary function: left above the key

`2nd[`*`e`*`]` **=** the Euler number ***e***

Third function: right above the key

`ALPHA[E]` **=** the variable **E**

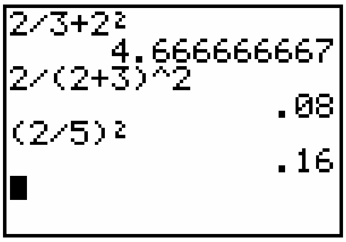


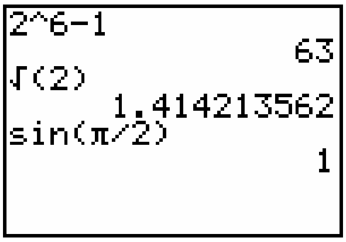
On these calculators you can only press one key at a time.

2 Elementary actions

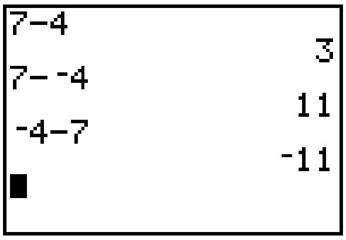
2.1 Calculation

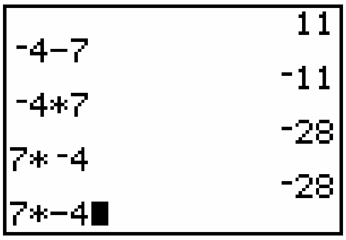
The way you calculate with the **TI-83/84 Plus** is the same as on a scientific calculator. To perform the calculations mentioned below you just press the keys in the same order you see the calculations on the home screen. You carry out your input (or a command) by pressing **ENTER**.



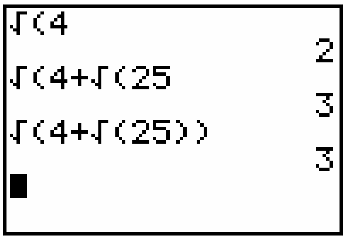


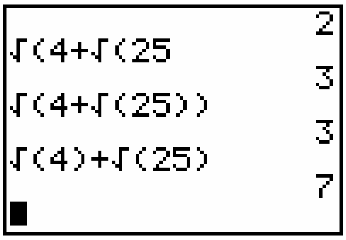
– = the difference
(-) = the opposite



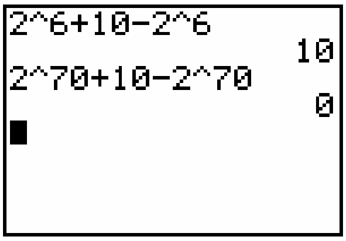


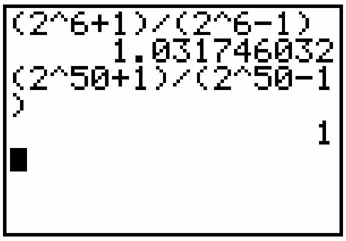
Notice that if you press √ or **SIN** automatically a first bracket appears. It's not necessary to place the second bracket but from a pedagogical point of view it's recommended.





The **TI-83/84 Plus** is a graphing calculator that does all its calculations numerically and in its standard mode it uses floating point numbers. Sometimes this gives strange results:





2nd[ENTRY] = recall previous performed calculations
CLEAR = clear the input line or the home screen

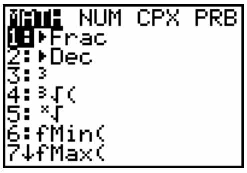
2.2 Menus

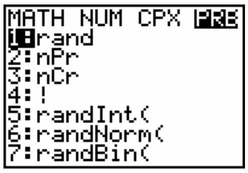
The advanced function keys contain menus. The use of menus we will explain by means of the **MATH** menu.

After pressing **MATH** the home screen will be replaced by the **MATH** menu.

◄ or ► : navigation between the submenus
▼ or ▲ : navigation in a submenu

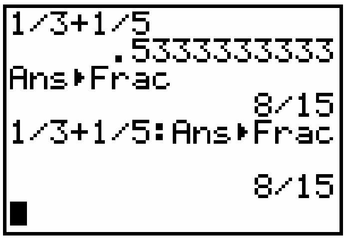
The **MATH** menu contains the following four submenus:

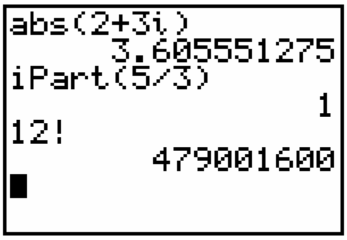


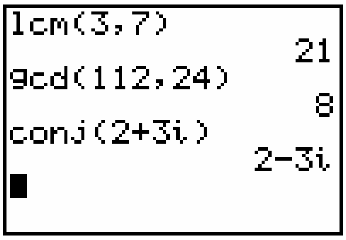


To select a command you can press the desired number (or letter) or you can mark the command and press **ENTER**.

Ans = the variable that contains the last result (**Last Answer - 2nd[ANS]**)







2nd[*i*] = the complex number *i*

You can always leave a menu without making a choice by pressing **CLEAR** or **2nd[QUIT]**.

2nd[QUIT] = return to home screen

Activity 1

Cross numbers

Fill in the following cross numbers. For each figure, decimal point and minus sign you need a separate box. Use an accuracy of two decimals (without rounding and as **MODE FLOAT**).

2nd[EE]6 = 10^6 or **2nd[10^x]** $\sqrt[3]{8} = 3\sqrt[x]{\ }\,8$ – **MATH<MATH> 5:** $\sqrt[x]{\ }$ or **3:** $\sqrt[3]{\ }($

Across

1. $\frac{463}{94}\cdot 47$

4. $4\left(\frac{458+\frac{1}{4}}{3}\right)$

8. $2^{10}-\sqrt{196}$

11. $10\left(\cos\left(\frac{\pi}{3}\right)+2^2\cdot 5\cdot\sin\left(\frac{\pi}{6}\right)\right)$

12. $2\left(307+\sqrt{96100}\right)10^3$

Down

2. $-(-81)^3$

4. $\frac{9710}{25\cdot 63}$

3. $\frac{(13.6\cdot 10^2)(2.8\cdot 10^{-4})}{24.3\cdot 10^{-4}}$

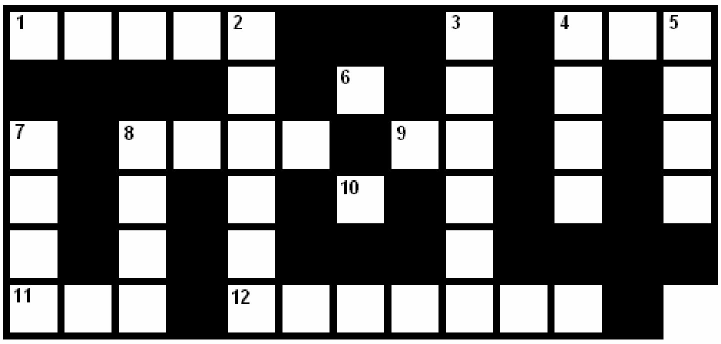
5. $\sqrt{765^2+1836^2}$

6. $\frac{2\cdot 10^{-2}}{63}\sqrt{500(17852+1993)}$

7. $6\sqrt{11}+\sqrt[3]{4.4}-1.83^5$

8. $\frac{12+3}{7+5}$

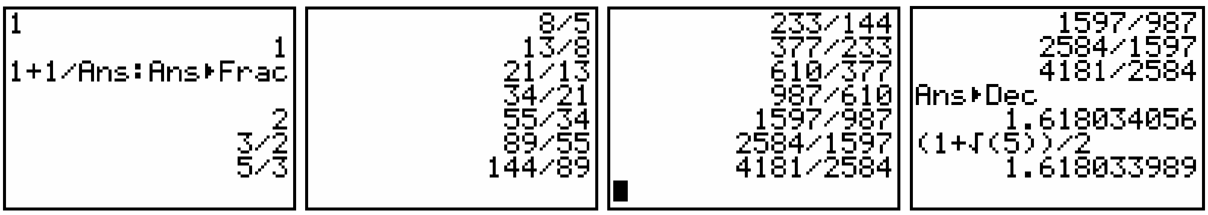
9. $2002-11^{-11}-2002$



Activity 2

The Golden Number φ

Investigate the behavior of the following sequence: $1,\ 1+1,\ 1+\frac{1}{1+1},\ 1+\frac{1}{1+\frac{1}{1+1}},\ 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}},\ \ldots\ldots$

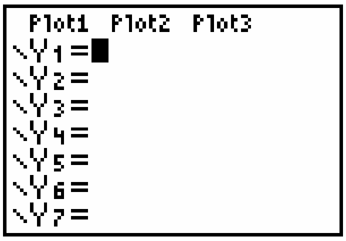


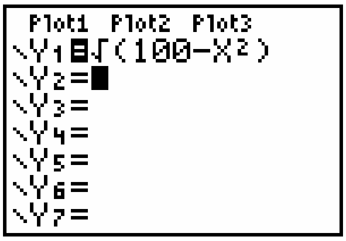
ENTER = Redo the last carried out command

2.3 Function Graphing

a. Definition of a function

Press **Y=** and define the variable **Y1** as mentioned below (Press **ENTER** after the input of the function).

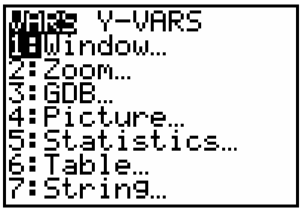


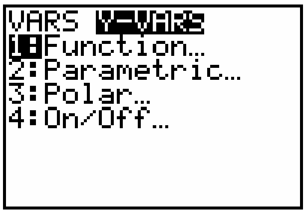


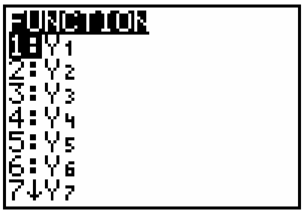
Notice that after the definition of the variable **Y1** the corresponding equal sign will be marked. This means that the function **Y1** is ready to be plotted. Then go on defining **Y2** as **-Y1**.

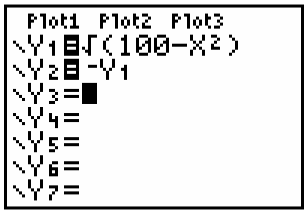
!! It is not possible to type in the variable **Y1** as **Y** followed by **1** !!

Select **Y1** out of the variable menu: **VARS<Y-vars> 1:Function...**





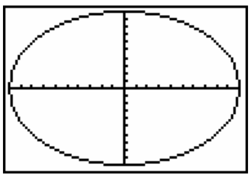




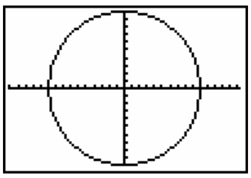
b. Graph of a function

After the definition of a function you can plot a function by selecting an appropriate window in the **ZOOM** menu.

Select first **6:Zstandard** and then **5:Zsquare**. Compare both results.



ZStandard



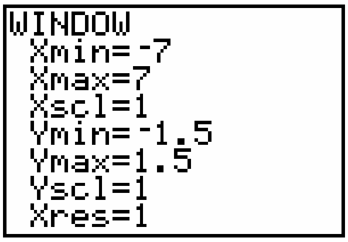
Zsquare

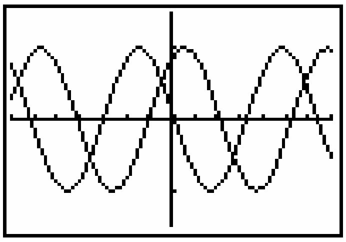
If you know the window settings you can plot the function immediately by pressing **GRAPH**.

You can turn off or deselect the functions **Y1** and **Y2** (not to be plotted) by placing the cursor on the equal signs and press **ENTER**.

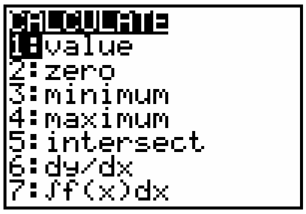
And to delete the function description place the cursor on the function description and press **CLEAR**.

You can also define the window settings manually by pressing the **WINDOW** key. Fill in the following setting and plot the function **sin(x+{1,3})** with these settings.





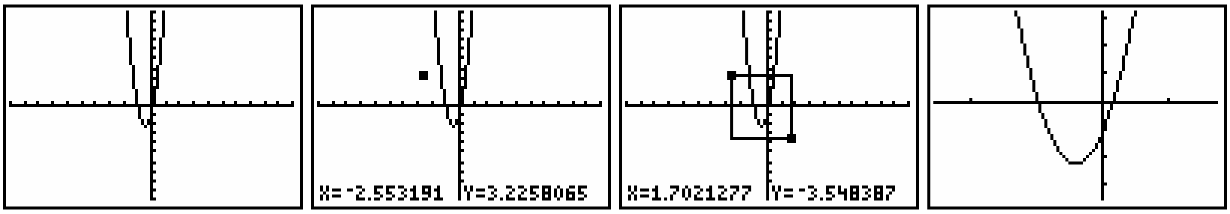
c. Numerical calculations



On the graph of a function several calculations can be done using the **CALC**-menu (**2nd[CALC]**). We will illustrate this by giving two examples:

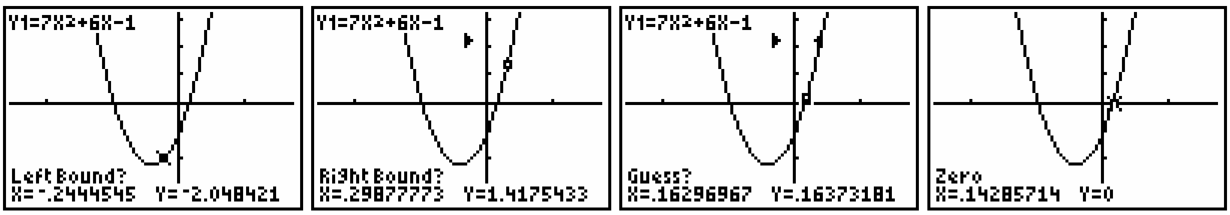
→ Determine the zeros of the function $f(x) = 7x^2 + 6x - 1$ ←

Below on the left you see the graph of the function f in a standard window. Use the **ZOOM** command **1:Zbox** to blow up the rectangle below. Put the cursor on the screen where you want to put the first vertex and press **ENTER**. Do the same for the opposite vertex. The enlargement will be shown automatically.



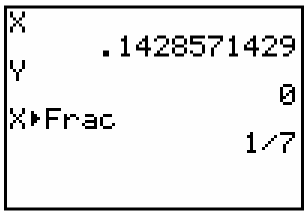
A continuous function with positive and negative values on a segment has a zero in that segment. We will calculate (approximately, numerically) a zero of f in such a segment with the **2nd[CALC] 2:zero**.

First enter the left and the right bound of the segment by moving the cursor to the desired value or by entering a numerical value and then pressing **ENTER**. Afterwards you need to enter a guess, the seed that will start the numerical process.



But a simple algebraic calculation gives us the zero $x = \frac{1}{7}$.

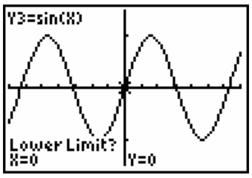
The calculator stores the coordinates of the zero in the variables **X** en **Y**. On the home screen you can try to transform the **X** coordinate into a rational number.

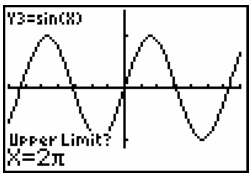


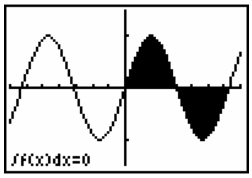
→ Definite integral ←

The command **2nd[CALC] 7:∫f(x)dx** calculates a numerical approximation of the definite integral.

$$\int_0^{2\pi} \sin(x)\,dx$$







Also here you can enter the lower and upper limit automatically or by selecting points on the graph.

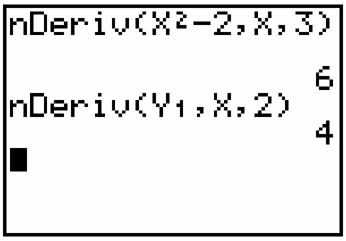
d. The derivative function

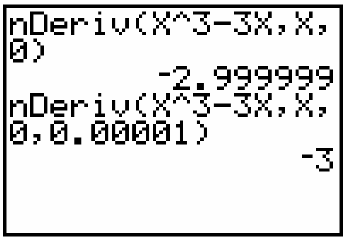
→ The numerical derivative ←

The command **MATH<MATH> 8:nDerive** calculates the numerical derivative of a function in a point: **nDerive**(expression.variable,value[,ε]). This command applies the following formula with the standard value $\varepsilon = 0.001$:

$$f\,'(x) = \frac{d\,f(x)}{dx} \approx \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon}\left(= \frac{\Delta y}{\Delta x}\right).$$

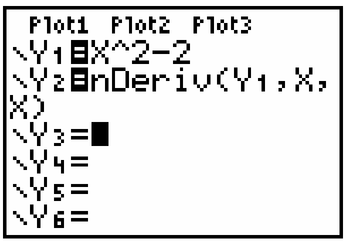
For **Y1=X^2-2** this command gives the following results. Study the right screen!

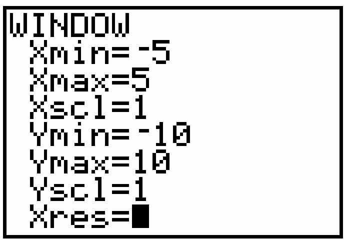


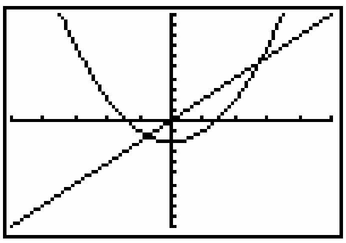


→ The derivative function ←

When you use the **nDerive** command in the **Y=** function definition window you can create a graphical representation of the derivatives of a lot of not too complicated, well-behaving functions. Define the functions **Y1** and **Y2** as mentioned below and plot both functions with the following window settings.







The plot above shows the relation of the decrease and increase of the function in terms of the sign of the derivative function.

Activity 3

An algebraic oddity

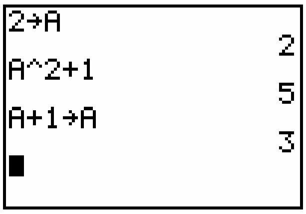
1. Define the function **Y1=X^3-5x^2+2x+10**.

2. Plot this function with the window settings **4:Zdecimal**.

3. Determine a linear function **Y2** so that the graphs of **Y1** and **Y2** have three intersection points.

4. Put the *x* coordinate of the intersection points into the variable **A**, **B** en **C**.

> **STO ▸** = store a value in a variable

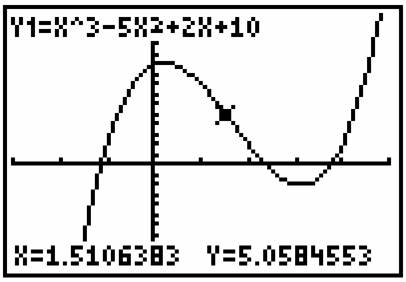


5. Compute the sum **A+B+C**.

6. Compare the results of all the students. Conclusion?

7. Explain algebraically!

> With the **TRACE** function you can move around on the graph.



To turn off the **TRACE** function you can press **GRAPH** or **CLEAR**.

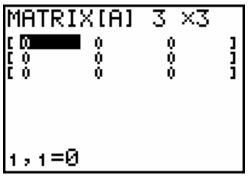
3 Matrices and systems of equations

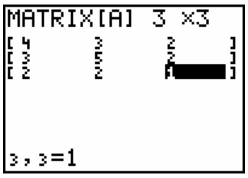
3.1 Matrices

a. Definition of a matrix

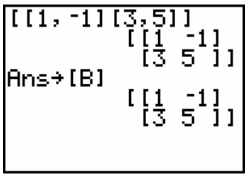
The easiest way to define a matrix is via the matrix editor, **MATRX<EDIT>**. You can define ten different matrices, **[A]** through **[J]**, on a **TI-83/84 Plus**. After the selection of the name, you can enter the dimension of the matrix. First enter the number of rows, followed by **ENTER** and then do the same for the numbers of columns. Automatically a matrix of the dimension entered appears with all its elements equal to zero.

For each element you can enter a number followed by **ENTER**. For example, after you have entered the first element the cursor will jump automatically to the next element to continue the entering. Navigation between the elements is possible by the arrow keys.





If you want to change the dimension or some elements after definition, reopen the matrix in the matrix editor.

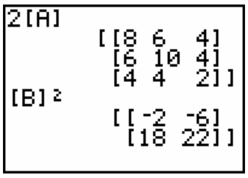


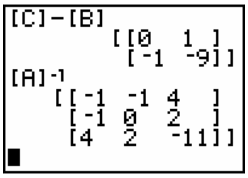
You can also define a matrix from the home screen as follows: **[[1,-1][3,5]]**. The use of matrices in expressions (see next section) and the assignment of a name to a matrix from the home screen has to be done via the sub menu **MATRX<NAMES>**.

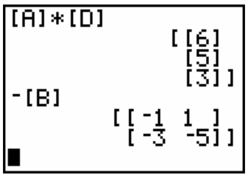
b. Calculations with matrices

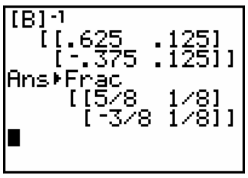
We will illustrate some elementary calculations with the following matrices:

$$A=\begin{bmatrix}4&3&2\\3&5&2\\2&2&1\end{bmatrix},\quad B=\begin{bmatrix}1&-1\\3&5\end{bmatrix},\quad C=\begin{bmatrix}1&0\\2&-4\end{bmatrix}\text{ and }D=\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

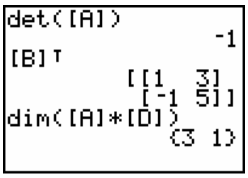


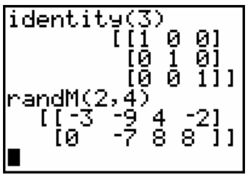






The submenu **MATRX<MATH>** contains several matrix functions. Some examples:

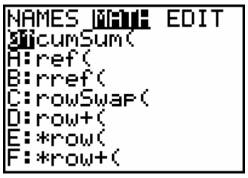


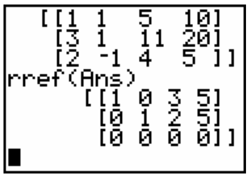


3.2 Systems of equations

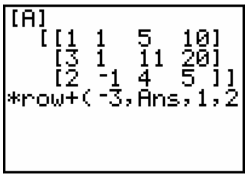
The submenu **MATRX<MATH>** contains functions by which we can solve systems of linear equations. An example:

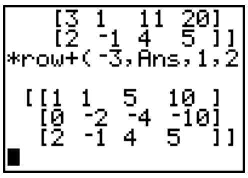
$$\begin{cases} x + y + 5z = 10 \\ 3x + y + 11z = 20 \\ 2x - y + 4z = 5 \end{cases}$$

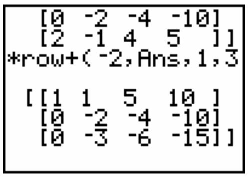


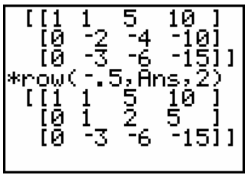


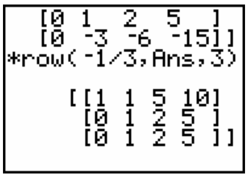
The command **B:rref(** generates the row reduced echelon form of a matrix (see example above) and **A:ref(** the reduced echelon form. With the commands **C** through **F** (elementary row operations) you can transform a matrix step by step into its reduced echelon form as shown below.

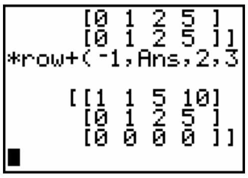










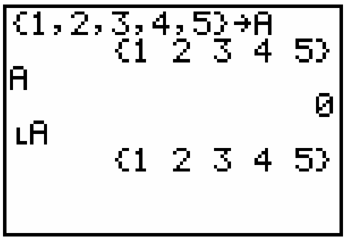


4 Lists

4.1 Defining lists

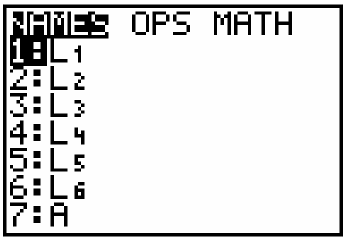
a. From the home screen

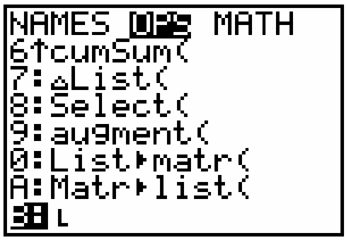
2nd[{] 1,2,3,4,5 2nd[}] STO▶ A



!! You can't use the letter **A** to work with list **A** !!

2nd[LIST]<NAMES> or **2nd[LIST]<OPS> B:ʟ A**

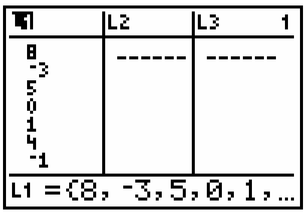


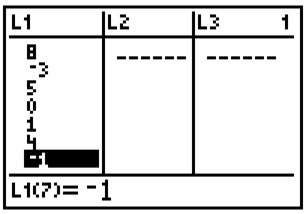


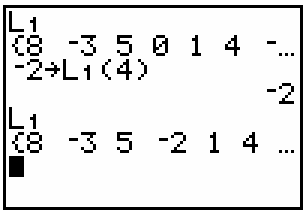
The **TI-83/84 Plus** has six standard lists in its memory: **L_1** through **L_6**. These lists are on the keys above the numerical keys **1** through **6**: **2nd[L_1], …, 2nd[L_6]**.

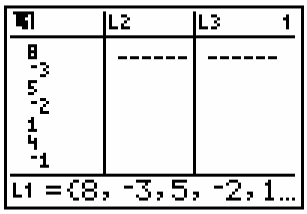
b. In the STAT editor - STAT 1:Edit

The **STAT** editor is a kind of worksheet in which you can enter data like in a spreadsheet.









DEL-key deletes a cell out of a list in the **STAT** editor

2nd[INS] inserts a cell (cell = **0**) into a list in the **STAT** editor

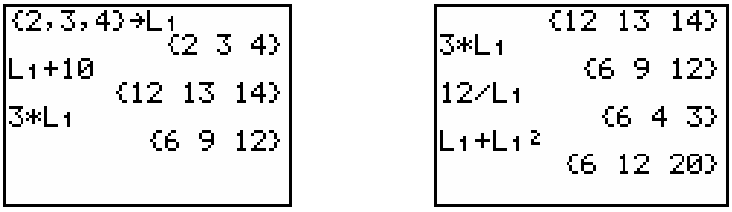
CLEAR clears the content of a list (with the cursor on the list name)

STAT 5:SetUpEditor = **STAT** editor with the lists **L_1** through **L_6**

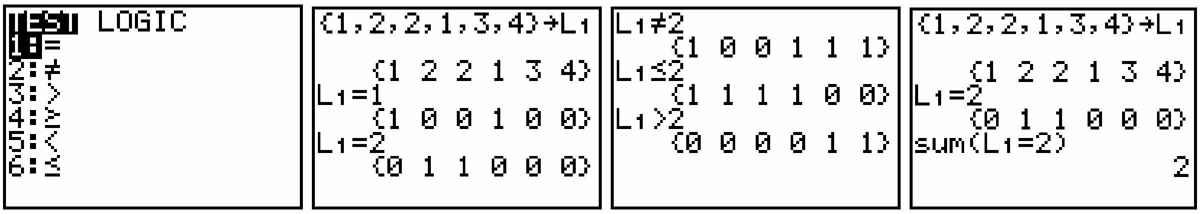
4.2 Calculations with lists

Define **L1** as {**2,3,4**} and do the following calculations:

L1+10, **3*L1**, **12/L1** and **L1 + L1**^{2}.



4.3 Logical operations with lists: 2nd[TEST]



With the command **2nd[LIST]<MATH> 5:sum(** in combination with a test you can examine how many elements meet a certain condition.

4.4 Operations with lists: 2nd[LIST]<OPS>

We will take a look at some of these commands.

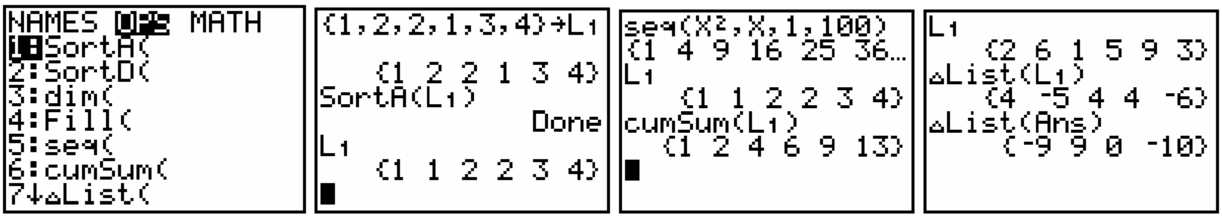
SortA(sorts a list in an **A**scending order

SortD(sorts a list in a **D**escending order

seq(generates a **seq**uence of numbers

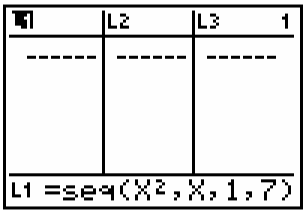
cumSum(determines the cumulative sums of the elements of a list

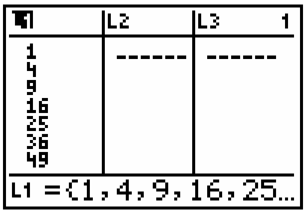
∆List(determines the difference of successive elements of a list



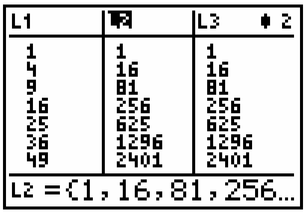
4.5 Lists and formulas

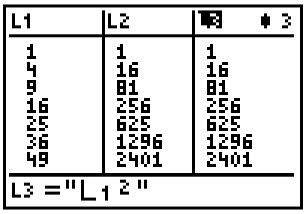
Define **L1** as **seq(X²,X,1,7)**.





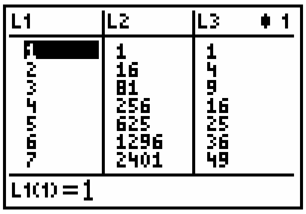
Define **L2** as **L1²** and **L3** as **ALPHA["] L1² ALPHA["]**.





⧫ = the formula is blocked

Change the content of **L1** into **seq(X,X,1,7)**. What happens with the content of **L2** en **L3**?



4.6 Scatter plot

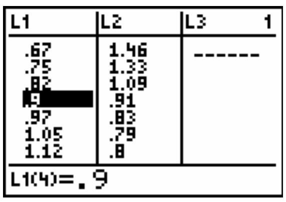
We will draw a scatter plot for the following data, the bounce of a ball, with **2nd[STAT PLOT] 1:Plot1**.

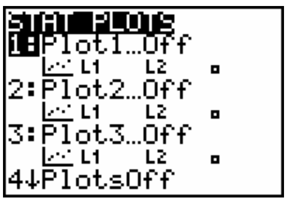
Put the cursor on **On** and press **ENTER**.

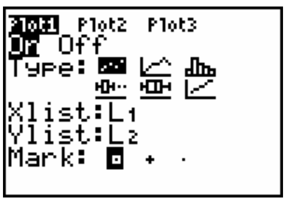
Select as **Type** the scatter plot icon ( ) and define Xlist as **2nd[L1]** and Ylist as **2nd[L2]**.

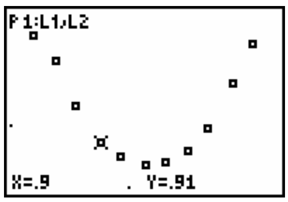
Plot the histogram with **ZOOM 9:ZoomStat**. With **TRACE** you can move the cursor from one data point to the next one.

L1 t	L2 $x(t)$
0.67	1.46
0.75	1.33
0.82	1.09
0.9	0.91
0.97	0.83
1.05	0.79
1.12	0.8
1.2	0.86
1.27	0.98
1.35	1.21
1.42	1.42









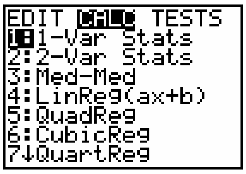
To connect the data points, select the xyLine icon ( ) instead of the scatter plot icon ( ).

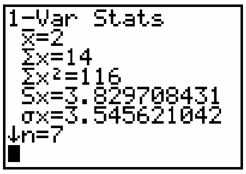
5 Descriptive statistics

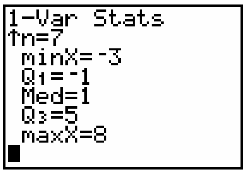
5.1 Statistical variables

To calculate the mean, median, standard deviation, … of one-variable data you use the command **1:1-Var Stats** of the **STAT<CALC>** menu.

After executing **STAT<CALC> 1:1Var Stats 2nd[L1]** (with **L1={8,-3,5,0,1,4,-1}**) the calculated values appear automatically on the screen. These values are stored into statistical variables (**VARS 5:Statistics…**).







5.2 Histogram

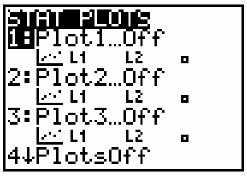
We will construct a histogram of the following data, the shoe size of 30 adult men, which we first put into list **L2**.

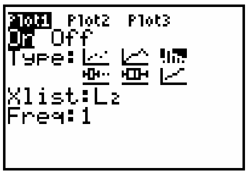
42	39	42	41	40	44	43	41	40	40
42	40	39	38	43	40	39	44	42	40
41	46	40	41	42	42	38	39	44	41

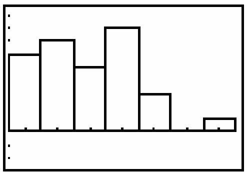
Start the construction with the command **2nd[STAT PLOT] 1:Plot1**.

Put the cursor on **On** and press **ENTER**. Select as **Type** the histogram icon (histogram icon) and define Xlist as **2nd[L2]**. The value of **Freq** is standard equal to **1**, which means that we work with the raw data.

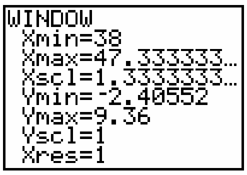
Plot the histogram with **ZOOM 9:ZoomStat**.

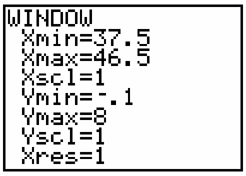






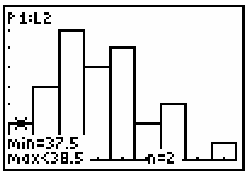
Take a look at the chosen window settings by pressing **WINDOW**. Set up the window as mentioned below and press **GRAPH**:

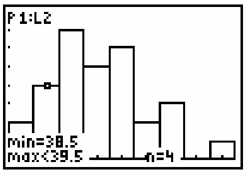


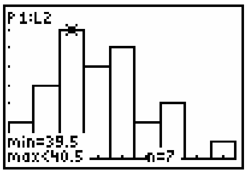


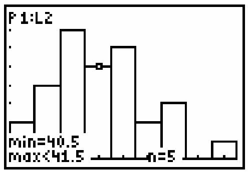
With these window settings the size of each class is 1 and shoe sizes are the middles of the classes.

With the **TRACE** function you can now determine the frequency of each shoe size. To turn off **TRACE** press **CLEAR** or **GRAPH**.

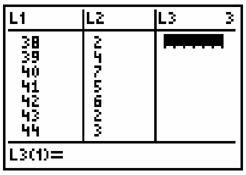


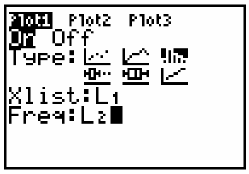




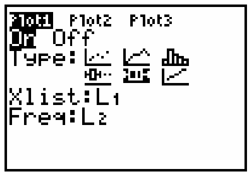


With these frequencies you can construct the following frequency table. To calculate the values of the statistical values and to plot the histogram based on the frequencies of the different data is done as follows.





5.3 Box plots

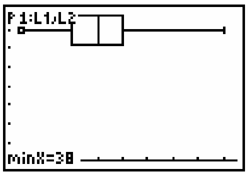


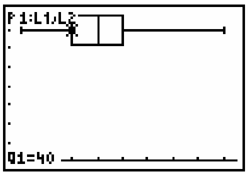
For the plotting of a box plot it is the same procedure as for the histogram.

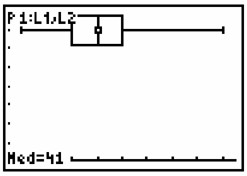
You can choose between a standard box plot ( ) and

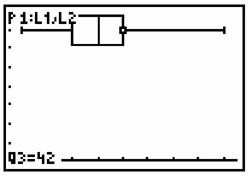
a modified box plot ( ).

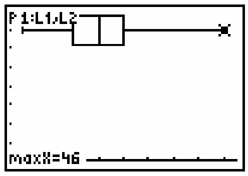
Using the **TRACE** function on a box plot produces the following five values: **minX**, **Q1**, **Med**, **Q3**, **maxX**.











5.4 Frequency table

The following program generates a frequency table of raw data in list **L1**. The different values of the data end up in **L2** and their frequency in **L3**. (How to program – See TI-83 Plus Guide – www.education.ti.com/guides)

```
PROGRAM:FREQTAB
SortA(L1)
ClrList L2,L3
1→I:1→J:1→T
While I ≤ dim(L1):L1(I)→L2(J)
  While L1(I)=L1(min({I+1,dim(L1)})) and I < dim(L1)
    I+1→I
    T+1→T
  End
T→L3(J)
J+1→J
1→T
I+1→I
End
```

6 Two distributions

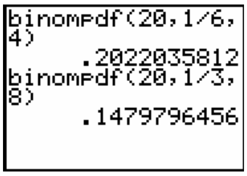
6.1 The binomial distribution

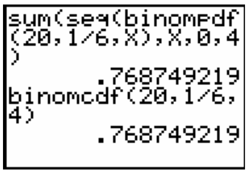
Suppose we throw a dice 20 times and the random variable *X* counts the number of times six spots appear. *X* has a binomial distribution with parameters *n* = 20, the number of trials, and $p = \frac{1}{6}$, the probability of success, six spots. The probability to have after 20 throws:

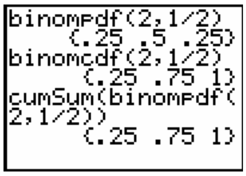
- exactly 4 times six spots, is: $\mathrm{P}(X=4) = \binom{20}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{16} = 0.202$,
- exactly 8 times "six or three": $\binom{20}{8}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^{12} = 0.148$,
- maximum 4 times a six: $\mathrm{P}(X \le 4) = \sum_{x=0}^{4}\binom{20}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{20-x} = 0.769$.

The **TI-83 Plus** can replace the distribution table or the calculation above by the following commands:

2nd[DISTR] 0:binompdf(and **2nd[DISTR] A:binomcdf(**





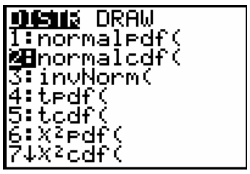


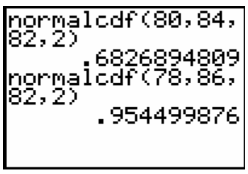
6.2 The normal distribution

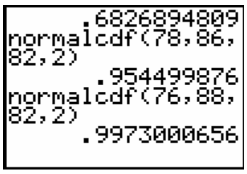
Suppose that the weight X of a group of students has a normal distribution with mean $\mu = 82$ kg and standard deviation $\sigma = 2$ kg – $X \sim N(82,2)$.

We will check the "68-95-99,7 rule" with the **normalcdf** command. This rule tells us that the surface under the graph of a normal distribution function between:

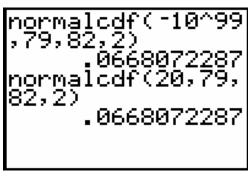
$\mu \pm \sigma$ is equal to 0,68269 ≈ 68%
$\mu \pm 2\sigma$ is equal to 0,95450 ≈ 95%
$\mu \pm 3\sigma$ is equal to 0,99730 ≈ 99,7%

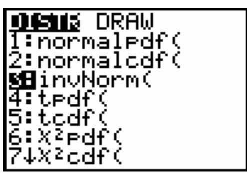


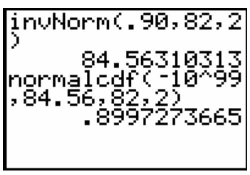




We will determine successively the percentage of students with a weight less than 79 kg and the weight below which 90% of the students' weights are situated. This weight is called the 90th percentile of the distribution.





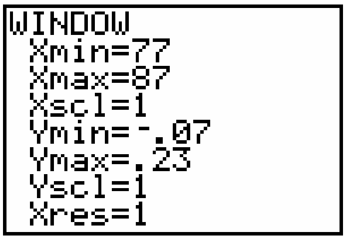


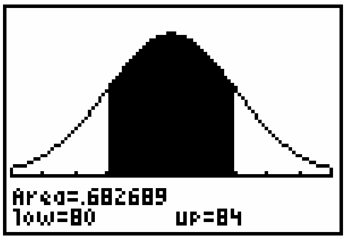
With the **normalcdf** command you can plot normal distribution functions.

Define **Y1** as **normalpdf(X,82,2)** and plot **Y1** with the window setting below.

It's also possible to shade a region under the graph of a normal distribution function.

2nd[DISTR]<DRAW> 1:ShadeNorm(80,84,82,2) gives you a graphical representation of the fact that the surface under the graph between $\mu \pm \sigma$ is equal to 68%.



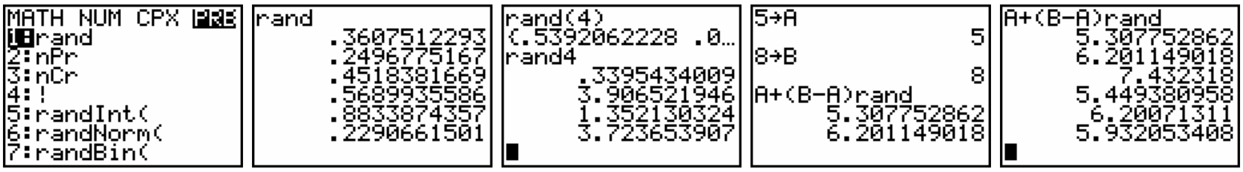


7 Simulation

7.1 Random numbers

The command **rand** generates the following random numbers:

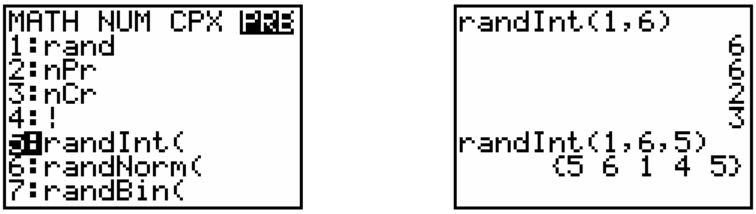
rand	→	an arbitrary number *x* between 0 and 1 ($0 < x < 1$)
rand(4)	→	a list of 4 arbitrary numbers between 0 en 1
rand4	→	an arbitrary number *x* between 0 en 4 ($0 < x < 4$)
A+(B-A)rand	→	an arbitrary number *x* between *A* en *B* ($A < x < B$)



The command **rand** starts generating at random numbers from a seed. The standard value of the seed is 0. Starting from a specific seed you will always get the same sequence of random numbers. So if you to generate at random numbers, by several students at the same time, it can be useful to let each student first choose an arbitrary value for the seed **rand**: e.g. **144→rand**.

randInt generates at random integers as follows:

randInt(1,6)	→	an arbitrary integer between 1 and 6 (1 and 6 included)
randInt(1,6,5)	→	a list of 5 arbitrary integers between 1 and 6



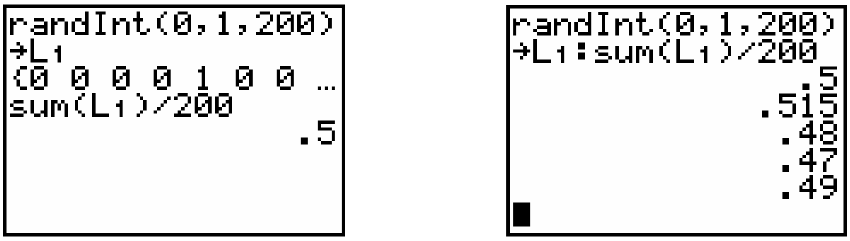
7.2 Coin tosses

To simulate the tossing of a coin we encode heads by **1** and tails by **0**.

We simulate two hundred coin tosses and we will save the results in list **L1** as follows:

randInt(0,1,200)→L1.

We can count the number of heads by the **sum** command. Dividing this result by 200 gives us the relative frequency of the event heads.



With the following list we will create a visualization of the previous simulation:

L1 = seq(X,X,1,200)

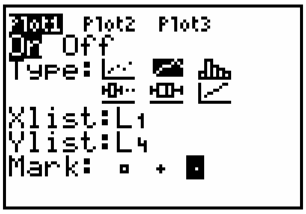
L2 = randInt(0,1,200)

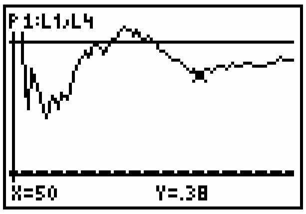
L3 = cumSum(L2)

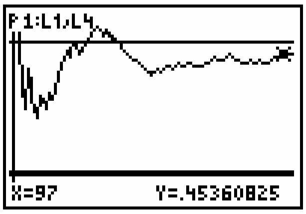
L4 = L3/L1

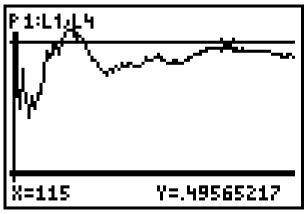
Y1 = 1/2

The following screens show that the relative frequency of the event heads in the long run approximate ½, the theoretical probability of the event heads.







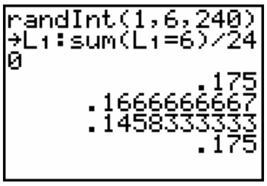


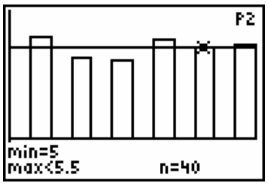
Notice that more tosses don't automatically cause a better approximation. If we continue tossing the relative frequency will get as close to ½ as we want. But no one can tell us how many tosses we have to do.

7.3 Throwing dice

We will simulate 240 throws of a dice by **randInt(1,6,240)→L1** and calculate the relative frequency of the events six spots as follows: **sum(L1=6)/240**.

The following histogram is a visualisation of Bernoulli's law of large numbers.

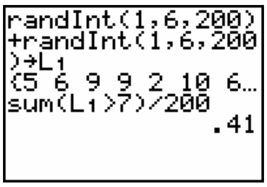


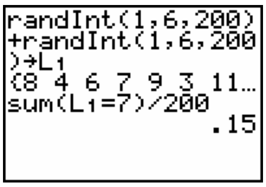


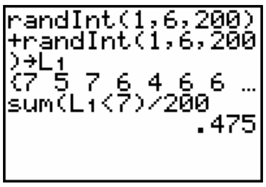
randInt(1,6)+randInt(1,6) simulates the throwing of two dice.

By repeatedly executing this command you can approximate the probability of the events *the sum of the point is less than, equal to or more than seven*.

Dice 1	Dice 2	Dice 1	Dice 2	Dice 1	Dice 2	Dice 1	Dice 2	Dice 1	Dice 2	Dice 1	Dice 2
1	1	2	1	3	1	4	1	5	1	6	1
1	2	2	2	3	2	4	2	5	2	6	2
1	3	2	3	3	3	4	3	5	3	6	3
1	4	2	4	3	4	4	4	5	4	6	4
1	5	2	5	3	5	4	5	5	5	6	5
1	6	2	6	3	6	4	6	5	6	6	6







8 An application: Linear programming

Linear programming is the branch of applied mathematics that deals with problems like the following example.

8.1 Apples and pears

Suppose you have € 3.6 for which you want to buy apples and pears. The price of one apple is € 0.2 and € 0.3 for a pear. How many apples and pears can I buy if you know that there are only 12 apples and 10 pears in the store?

Solution

Let's x represent the number of apples and y the number of pears.

Obvious the following conditions count: $x \geq 0$ and $y \geq 0$.

And there are the following constraints for x and y:

$20x + 30y \leq 360$, $x \leq 12$ and $y \leq 10$.

To solve the problem we need to find all the points (x, y) that satisfy:

$$\begin{cases} 20x + 30y \leq 360 \\ x \leq 12 \\ y \leq 10 \\ x \geq 0, y \geq 0 \end{cases}$$

We try to solve the problem by a graphical approach by plotting the linear relations $20x + 30y = 360$, $x = 0$, $x = 12$, $y = 10$ and $y = 0$.

Therefore we define the functions: **Y1=12 - $\frac{2}{3}$X**,

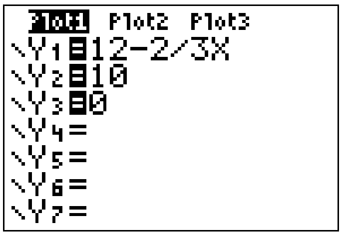
Y2=10,

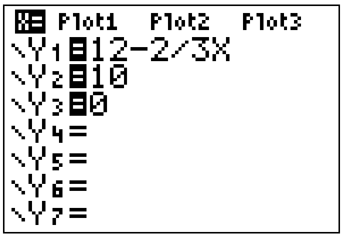
Y3=0,

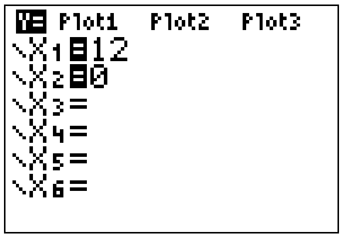
X1=12,

X2=0.

To define **X1=12** and **X2=0** you first need to activate the application *Inequality Graphing* [1]. And then select **X=**.

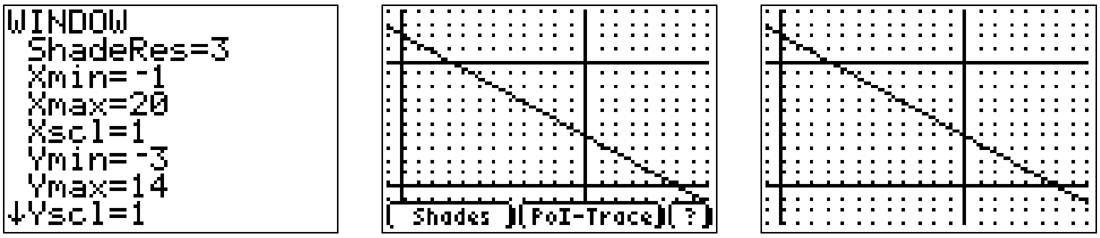




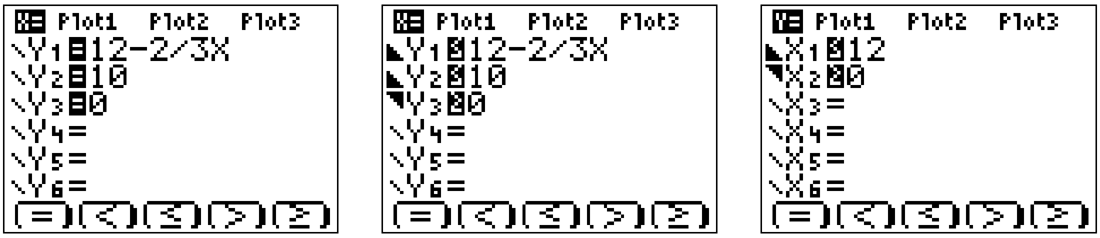


[1] See 9. Appendix.

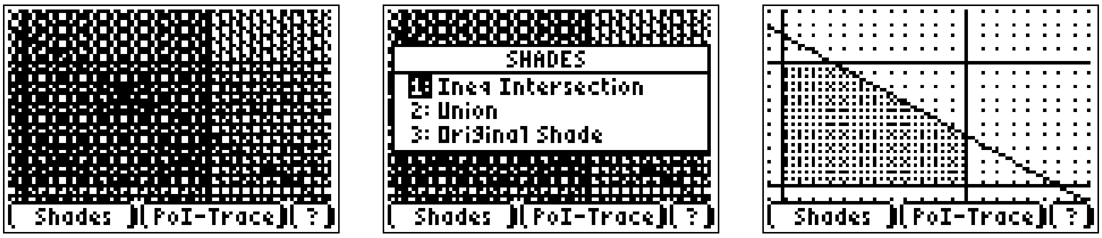
These definitions result into the following graph (press **TRACE ClEAR** to remove the menu at the bottom of the screen):



All the points in the enclosed area are solutions for our problem. It's possible to shade this area and to calculate its vertices. To shade we need to change the equal signs with **F1** through **F6** as follow:

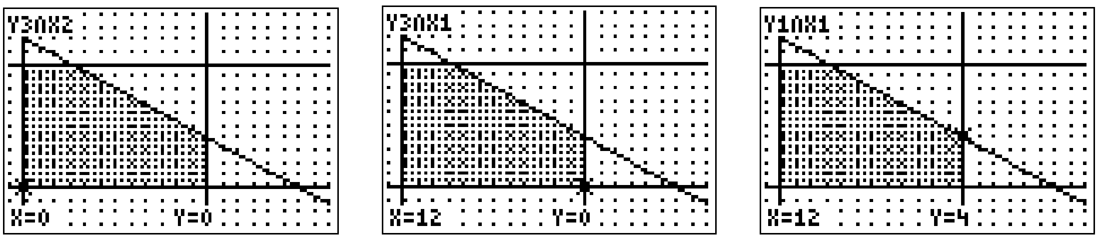


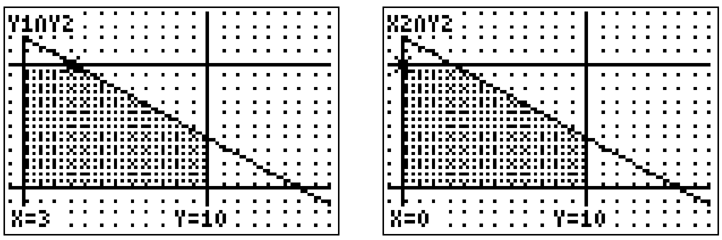
Press **GRAPH**, then select **Shades** and **1: Ineq Intersection**.



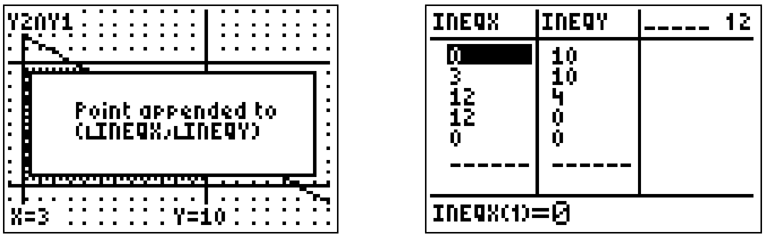
We will now calculate the vertices of shades with **PoI-Trace**:

◀ ▶ = change the first function ▲ ▼ = change the second function

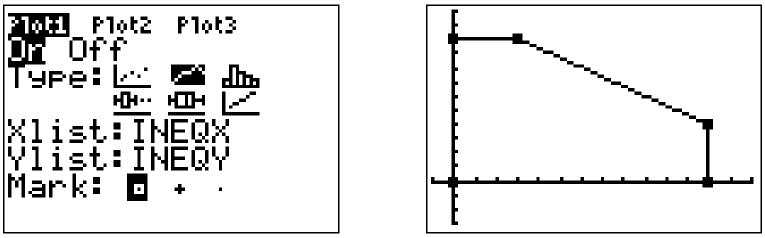




You can store a selected vertex by pressing **STO ▸**. The coordinates of the vertex will automatically be stored in the lists **INEQX** and **INEQY**.



With these lists it's still possible to plot the area even after quitting *Inequality Graphing* and/or deleting the functions. On the graph below the grid is turned off.

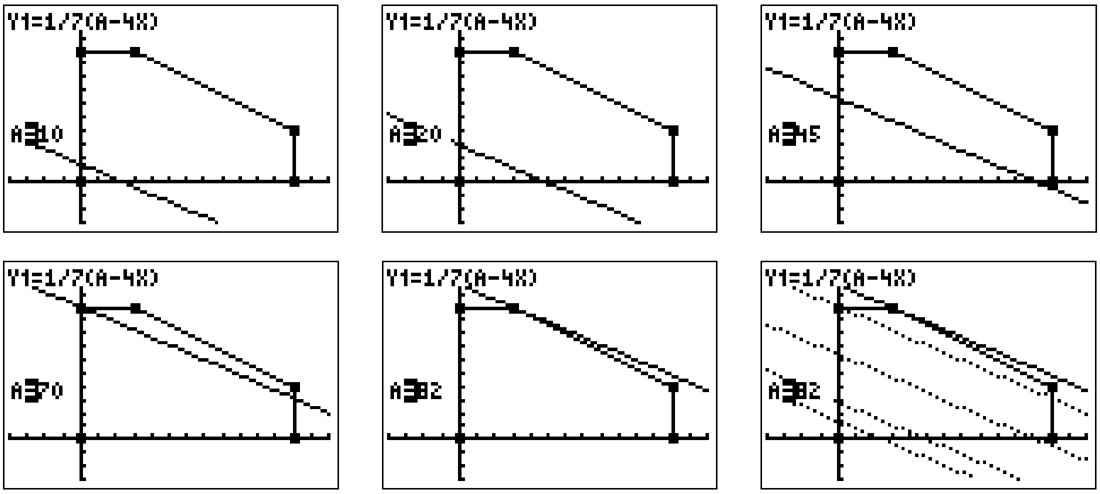


Let's make our example a little bit more complicated. We want to have an as high as possible content of vitamin C in our purchase. Suppose on apple contains 4 gram of vitamin C and a pear 7 gram.

To solve this problem we need to find the maximum value of $4x+7y$ over the area determined above.

To investigate this problem graphically we define the parameter A as follow $A=4x+7y$ and the function **Y1** = $\frac{1}{7}(A-4x)$.

Activate the application *Transformation Graphing* [2] (deactivate first *Inequality Graphing*) and investigate the value of A for several points in the enclosed area.

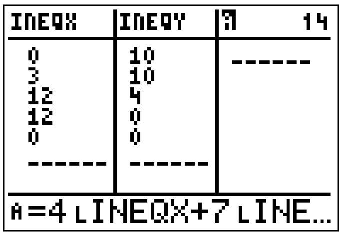


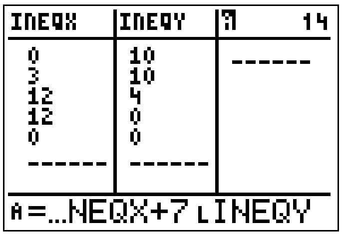
TrailOn – 2nd[FORMAT]

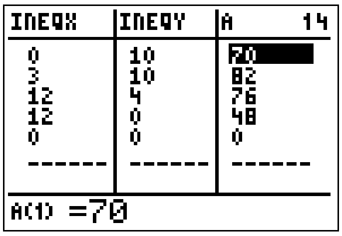
When we study the variation of the parameter A we see that the maximum value of A will be found in one of the vertices.

[2] See 9. Appendix

With **STAT 1:Edit…** we can calculate these values for A as follows:







The maximal amount of vitamin C is 82 gram with a purchase of 3 apples and 10 pears.

8.2 The simplex method

We can write the previous example as follows:

Maximize $4x+7y$

Subject to $20x+30y\le 360$ (1)

$x\le 12$

$y\le 10$

$x\ge 0, y\ge 0$

The simplex method always starts from a feasible solution. For our use we will take the origin $x=0$ and $y=0$. Of course these x and y values aren't the ones that gives us the maximum value for $4x+7y$.

We will rewrite the inequalities into equalities by introducing three new variables u,v,w; called slack variables:

$u=360-20x-30y\le 360$

$v=12-x$

$w=10-y$

We define $z=4x+7y$. The old variables x and y are called the decision variables.

So now we can rewrite our problem as follows:

Maximize $z=4x+7y$

Subject to $u=360-20x-30y$ (2)

$v=12-x$

$w=10-y$

$x\ge 0, y\ge 0, u\ge 0, v\ge 0, w\ge 0$

<u>Note</u>

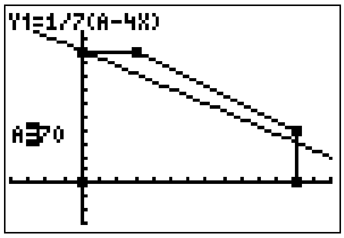
- Each feasible solution of (1) can be extended to a feasible solution of (2).
- Each feasible solution of (2) can be restricted to a feasible solution of (1).
- Each optimal solution of (1) corresponds with an optimal solution of (2).

Our feasible solution to start from is $x=0, y=0, u=360, v=12, w=10$. (3)
This solution gives $z=0$.

We will try to find successive improvements out of this feasible solution x,y,u,v,w to end with a maximal solution. This means that out of x,y,u,v,w we try to deduce a feasible solution $\tilde{x},\tilde{y},\tilde{u},\tilde{v},\tilde{w}$ with $4\tilde{x}+7\tilde{y}\ge 4x+7y$.

If we look at $z = 4x + 7y$ we see that if we increase y, z will increase faster as when we increase x. So we will increase y and keep $x = 0$. How much can we increase y?

For $x = 0, u \geq 0, v \geq 0, w \geq 0$ the following constraints count:



$$\begin{cases} 360 - 30y \geq 0 \\ 12 \geq 0 \\ 10 - y \geq 0 \end{cases} \Leftrightarrow \begin{cases} y \leq 12 \\ 12 \geq 0 \\ y \leq 10 \end{cases} \Rightarrow \quad y \leq 10\,.$$

In other words y can increase up to 10. So we become our next solution: $x = 0, y = 10, u = 60, v = 12, w = 0$ which yields $z = 70$.

In our next step we are going for an ever better feasible solution. How can we do this?

We need to manufacture a new system of linear constraint to continue. If we look at (2) we see that it expresses the variables u, v, w that assume positive values in (3) in terms of those variables x, y that assume zero. And also z is expressed in (2) in terms of x, y.

Note that y changed its value from zero to positive and w from positive to zero. So we need to change their position in the system of equations, from the right-hand side to the left-hand side and vice versa. We call y the entering variable and w the leaving variable.

We start with the newcomer y on the left-hand side. With the third equation of (2) we can express y in terms of x, w: $w = 10 - y \Leftrightarrow y = 10 - w$.

Next we express u, v and z in terms of x, w

$$u = 360 - 20x - 30y = 360 - 20x - 30(10 - w) = 60 - 20x + 30w$$
$$v = 12 - x$$
$$z = 4x + 7y = 4x + 7(10 - w) = 70 + 4x - 7w$$

So we can rewrite our problem as follows:

Maximize $\quad z = 70 + 4x - 7w$

Subject to $\quad u = 60 - 20x + 30w$

$v = 12 - x$

$y = 10 - w$

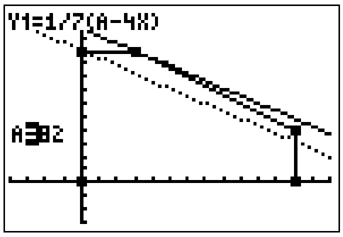
$x \geq 0, y \geq 0, u \geq 0, v \geq 0, w \geq 0$

From our second feasible solution $x = 0, y = 10, u = 60, v = 12, w = 0$ with $z = 70$ we will again try to find an improvement.

If we look at $z = 70 + 4x - 7w$ the only way to let increase z is to increase x.

How much can we increase x?

For $w = 0, y \geq 0, u \geq, v \geq 0$ the following constraints count:



$$\begin{cases} 60 - 20x \geq 0 \\ 12 - x \geq 0 \\ 10 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \leq 3 \\ x \leq 12 \\ 10 \geq 0 \end{cases} \Rightarrow \quad x \leq 3$$

In other words x can increase up to 3 and our next feasible solution is: $x = 3, y = 10, u = 0, v = 9, w = 0$ with $z = 82$.

Now we express all variables and z in terms of u, w. Again we will start with the newcomer x:

$$u = 60 - 20x + 30w \Leftrightarrow 20x = 60 - u + 30w \Leftrightarrow x = 3 - \frac{1}{20}u + \frac{3}{2}w\,.$$

It follows that:

$$v = 12 - x = 9 + \frac{1}{20}u - \frac{3}{2}w$$

$$y = 10 - w$$

$$z = 70 + 4x - 7w = 70 + 4(3 - \frac{1}{20}u + \frac{3}{2}w) - 7w = 82 - \frac{1}{5}u - w$$

When we look at z it's clear we can not increase z anymore by increasing u or w.

This means we found an optimal solution $z = 82$ for $x = 3$ and $y = 10$.

The method we just used to find an optimal solution is called the simplex method. In this particular example x and y has to be integers but everything stays the same if we consider x and y as real variables.

8.3 The simplex method using matrices

We rewrite our example into the following modified form.

$$\begin{array}{l} 20x + 30y + u = 360 \\ x + v = 12 \\ y + w = 10 \\ \hline -z + 4x + 7y = 0 \end{array} \quad \text{or} \quad \begin{array}{cccccccccccc} & & 20x & + & 30y & + & u & & & & = & 360 \\ & & x & & & & & + & v & & = & 12 \\ & & & & y & & & & + & w & = & 10 \\ \hline -z & + & 4x & + & 7y & & & & & & = & 0 \end{array}$$

Using only the coefficients we can use the following matrix to represent our example.

$$\begin{pmatrix} 20 & 30 & 1 & 0 & 0 & 360 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 7 & 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Step 1</u>

Examine the elements of the last raw, except the one in the last column (which represent the present value of -z. If all the elements or negative, the matrix represent an optimal solution. Otherwise select the column associated with the largest positive number. This column is called the pivot column and corresponds with the entering variable.

$$\begin{pmatrix} \boxed{20} & 30 & 1 & 0 & 0 & 360 \\ \boxed{1} & 0 & 0 & 1 & 0 & 12 \\ \boxed{0} & 1 & 0 & 0 & 1 & 10 \\ \boxed{4} & 7 & 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Step 2</u>

We will calculate the ratios $\frac{p}{q}$ of the elements p of the rightmost column and the positive elements q of the pivot column (except for the last column). If they our all negative the problem is unbounded (see further).

The row with the smallest ratio $\frac{p}{q}$ is called the pivot row and corresponds with the leaving variable.

$$\begin{pmatrix} 20 & 30 & 1 & 0 & 0 & 360 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 7 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \longrightarrow \frac{360}{30}=12 \\ \\ \longrightarrow \frac{10}{1}=10 \\ \\ \end{matrix}$$

Step 3

In this step we divide each element of the pivot row with the pivot (= intersection of the pivot column and the pivot row). In our case (pivot = 1) we don't need to do anything.

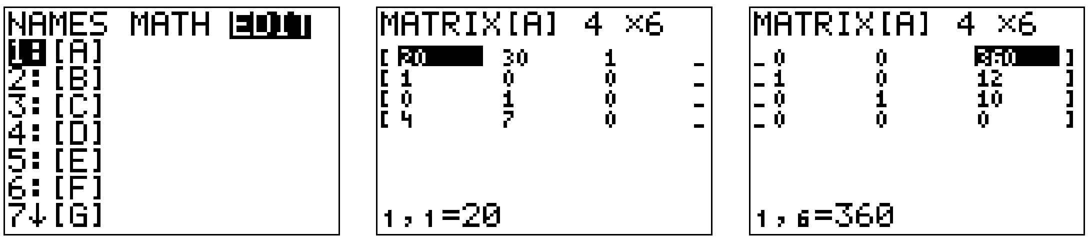
It's not a bad idea to add a column with the positive variables of our present solution.

$$\begin{pmatrix} 20 & 30 & 1 & 0 & 0 & 360 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 7 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} u \\ v \\ w \\ \\ \end{matrix}$$

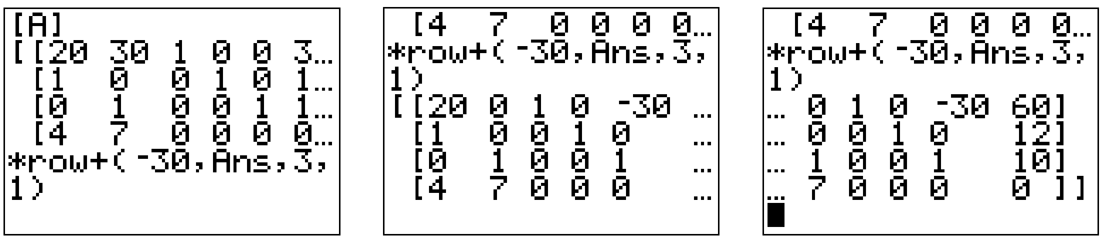
Step 4

Use elementary row operations (**2nd[MATRIX]<MATH>**) to make all the elements of the pivot column, except the pivot, zero.

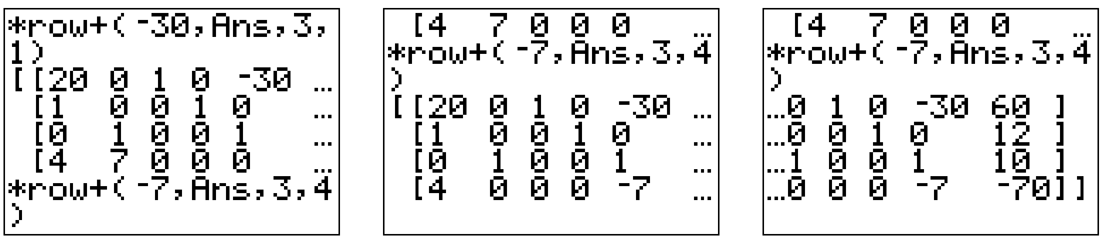
Input of the matrix



$-30\ R_3 + R_1$



$-7\ R_3 + R_4$



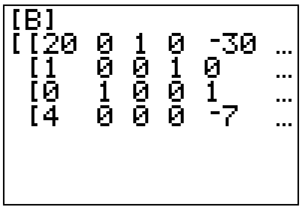
So we become the following new matrix, with $z = 70$ and $x = 0, y = 10, u = 60, v = 12$ and $w = 0$:

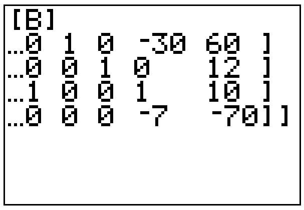
$$\begin{pmatrix} 20 & 0 & 1 & 0 & -30 & 60 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 0 & 0 & 0 & -7 & -70 \end{pmatrix} \begin{matrix} u \\ v \\ y \\ \end{matrix}$$

We need to redo the previous four steps, starting from this matrix, to find a better feasible solution.

Step 1 & 2

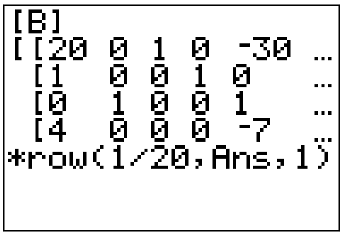
$$\begin{pmatrix} \boxed{20} & 0 & 1 & 0 & -30 & 60 \\ 1 & 0 & 0 & 1 & 0 & 12 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 4 & 0 & 0 & 0 & -7 & -70 \end{pmatrix} \begin{matrix} u \\ v \\ y \\ \end{matrix}$$

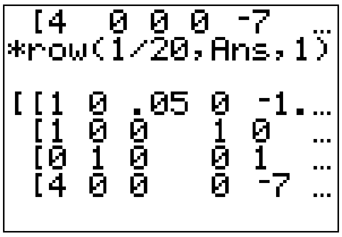


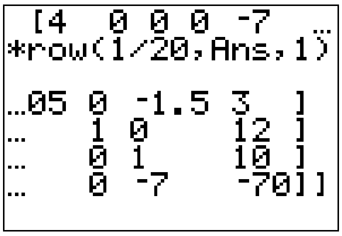


Step 3 & 4

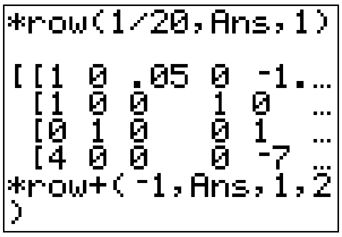
$-\frac{1}{20} R_1$

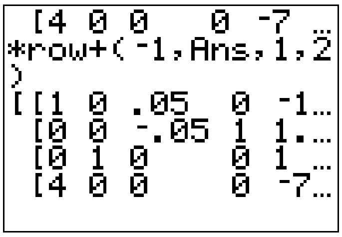


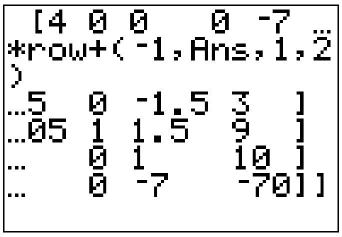




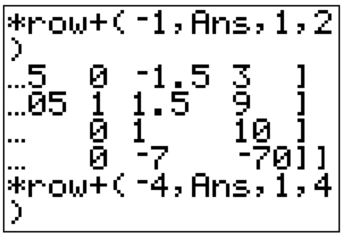
$-R_1 + R_2$

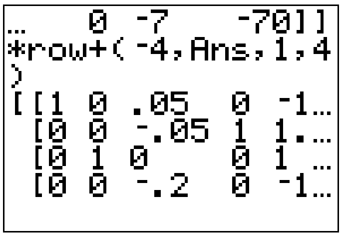


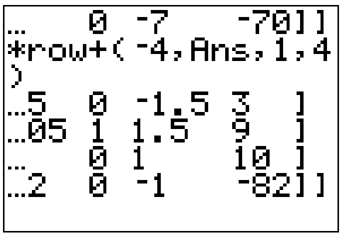


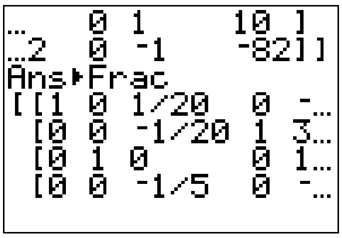


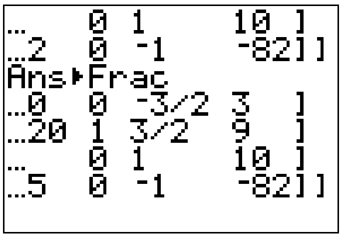
$-4 R_1 + R_4$











Our new matrix is: $\begin{pmatrix} 1 & 0 & \frac{1}{20} & 0 & -\frac{3}{2} & 3 \\ 0 & 0 & -\frac{1}{20} & 1 & \frac{3}{2} & 9 \\ 0 & 1 & 0 & 0 & 1 & 10 \\ 0 & 0 & -\frac{1}{5} & 0 & -1 & -82 \end{pmatrix} \begin{matrix} x \\ v \\ y \\ \\ \end{matrix}$

The last row of our matrix contains only negative numbers which means we reached an optimal solution $x = 3, y = 10, u = 0, v = 9, w = 0$ with $z = 82$.

8.4 Always a unique solution?

Without giving a complete discussion we will end with two examples to show that there is not always a unique solution.

(i) Several solutions – infinite many

Maximize $z = 2x + 4y$ subject to $x - y \le 2$, $x + 2y \le 16$, $x \ge 0, y \ge 0$

or

Maximize $z = 2x + 4y$ subject to $u = 2 - x + y$, $v = 16 - x - 2y$, $x \ge 0, y \ge 0, u \ge 0, v \ge 0$

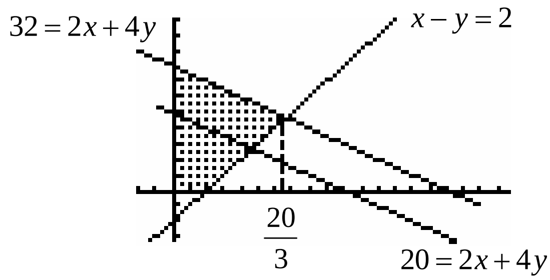
The second constraint give already an indication that the line which represent $2x + 4y - z = 0$ is parallel to one side of the area enclosed by the constaints.
In a following step we become:

Maximize $z = 32 - 2v$
subject to $y = 8 - 0.5x - 0.5v$
$u = 10 - 1.5x - 0.5v$
$x \ge 0, y \ge 0, u \ge 0, v \ge 0$

For each optimal solution ($z = 32$) counts $v = 0$, but not necessary $x = 0$. The condition for x is $10 - 1.5x \ge 0 \Leftrightarrow x \le \frac{20}{3}$.

For all x in $\left[0, \frac{20}{3}\right]$ we find an optimal solution $x, y = 8 - 0.5x, u = 10 - 1.5x, v = 0$.

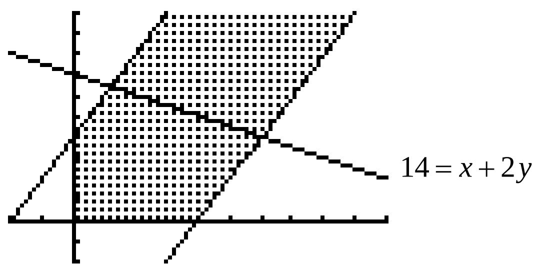
Note: $y = 8 - 0.5x \Leftrightarrow x + 2y = 16 \Leftrightarrow 2x + 4y = 32$.



(ii) No solution – an unbounded problem

Maximize $z = x + 2y$

subject to $-2x + y \le 4$
$2x - y \le 8$
$x \ge 0, y \ge 0$



9 Appendix

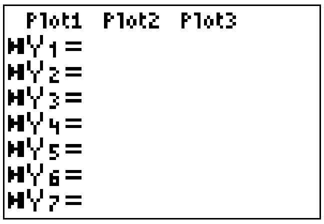
Graphing Calculator Software Applications (Apps) are pieces of software that you can download onto your calculator as you would add software to a computer to enhance its capabilities. Apps not only allow you to customize your TI calculator to meet your class needs, but also to upgrade it from one year to the next.

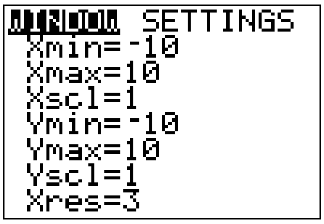
You can download Apps, as well as detailed guidebooks, for free from www.education.ti.com, Downloads.

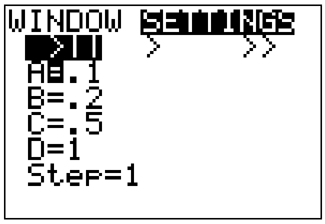
9.1 Transformation Graphing

Transformation Graphing allows visualizing dynamically how changes in a function's parameters effect its graph. This application enables students to discover several properties in terms of a function's parameters: roots, increasing and decreasing, symmetry, period, … It can also be used for modelling by manipulating coefficients to fit equations to data points.

Transformation Graphing is an application that once it's started it keeps running in the background. It changes the **Y=** window as follows and adds the menu **SETTINGS** to the **WINDOW** screen.







To quit Transformation Graphing you need to activate it again in the **APPS** menu and then select **1: Uninstall**. Note that it is not possible to run Transformation Graphing and Inequality Graphing at the same time.

With Transformation Graphing is possible to observe the effects of changing parameter values on the graph without leaving the graph screen. It is only available in the function mode and when it's active it's only possible to plot one function.

Transformation Graphing allows the use of four parameters: A, B, C, and D. All the others act like constants, using the value in the RAM memory.

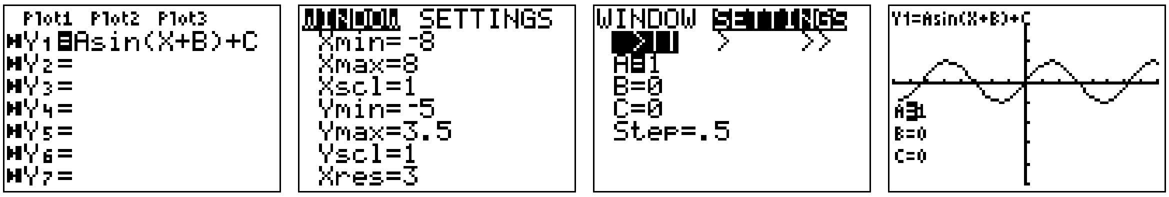
Transformation Graphing has three play types.

PLAY-PAUSE (>||) lets you change the parameter and plot the graph.

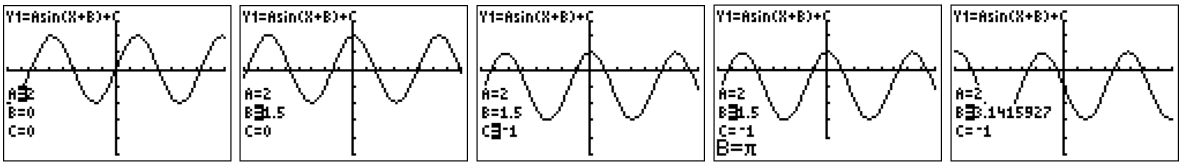
PLAY (>||) stores a series of changes and shows the corresponding graphs in a continuous slide show.

PLAY-FAST (>||) stores a series of changes and shows the corresponding graphs in a fast continuous slide show.

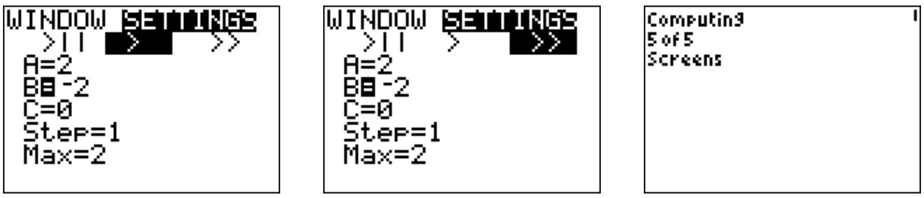
We will use the function $f(x) = A\sin(Bx) + C$ to illustrate how Transformation Graphing works. We will start with the following **WINDOW** settings.

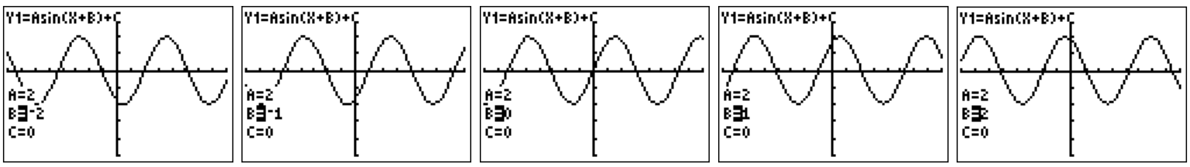


PLAY-PAUSE (>||)
Press ◄ ► to change the selected parameter and ▲ ▼ to select a different parameter. The graph will change automatically. It is also possible to enter a value manually. Select the parameter, enter the value and press **ENTER**.



PLAY (>||) and **PLAY-FAST** (>||)
With these options you can define a slide show per parameter. By putting the cursor on the equality sign and pressing enter you can select another parameter. Press **GRAPH** to start generating the screens for the slide show. The definitions below will generate 5 screens for the parameter B: from -2 to 2 in steps of size 1.

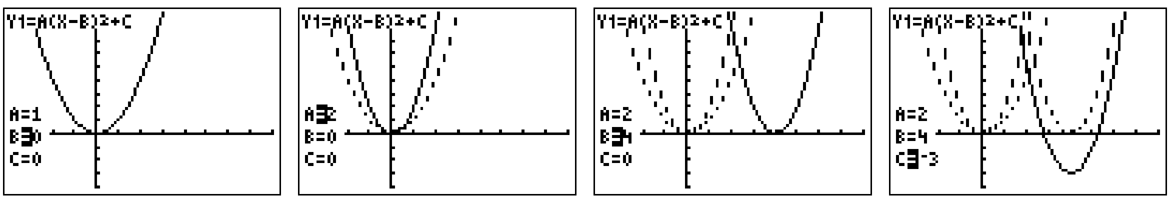




Press **ENTER** to pause the show and again to resume it and press and hold **ON** to stop.

Transformation Graphing also adds an extra setting to the graph format screen, **2nd[FORMAT]**: **TrailOff** or **TrailOn**.

With **TrailOn** you will see better the effect of changing a parameter because the previous graphs stay on the screen in a dotted format.

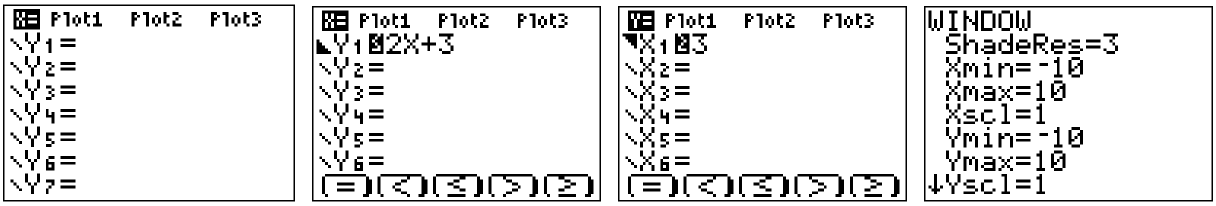


9.2 Inequality Graphing

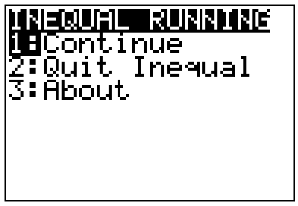
Inequality Graphing enables to enter inequalities using symbols, even inequalities involving vertical lines in an **X=** editor. It is possible to plot the inequalities, including union and intersection shades, and to store the intersection points between the corresponding functions.

With this application it is possible to add very easily a graphical approach to solving systems of linear equations (two variables) and to linear programming.

Inequality Graphing is an application that once it is started it keeps running in the background. It changes the **Y=** window as follows and adds an **X=** editor to it. It also adds a shade resolution item (ShadeRes) to the **WINDOW** settings.



To quit Inequality Graphing you need to activate it again in the **APPS** menu and then select **2: Quit Inequal**. Note that it is not possible to run Inequality Graphing and Transformation Graphing at the same time.

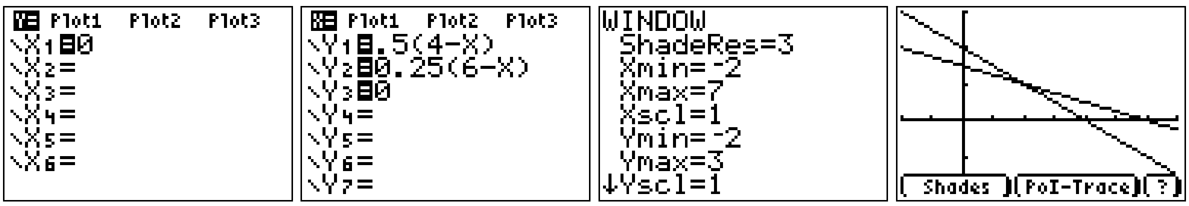


The following two examples will show how Inequality Graphing works.

Example 1
We will determine the region of points (x, y) that satisfy:

$$\begin{cases} x+2y \le 4 \\ x+4y \le 6 \end{cases} \text{ and } \begin{cases} x \ge 0 \\ y \ge 0 \end{cases}.$$

Therefore we define the linear functions **Y1=0.5(4-X)**, **Y2=0.25(6-X)**, **Y3=0**, **X1=12** and plot them with the following **WINDOW**-settings (press **TRACE ClEAR** to remove the menu at the bottom of the screen).



All the points in the enclosed area are solutions to our problem. It is possible to shade this area and to calculate its vertices.
To shade this area put the cursor on the equality signs to change them as follows into inequalities:

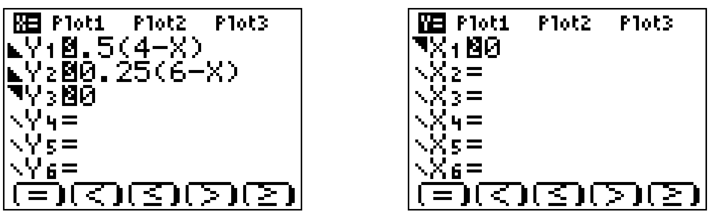
ALPHA F1 → =

ALPHA F2 → <

ALPHA F3 → ≤

ALPHA F4 → ≥

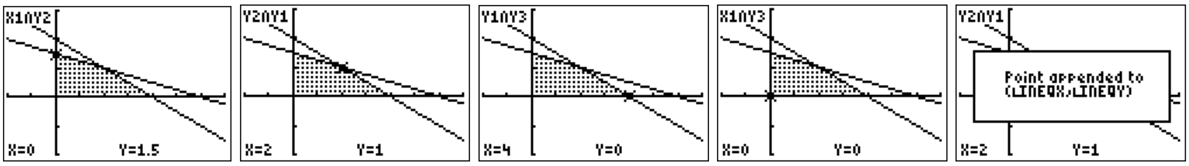
ALPHA F5 → >



Press **GRAPH**, select **Shades (ALPHA F1)** and **1: Ineq Intersection**.

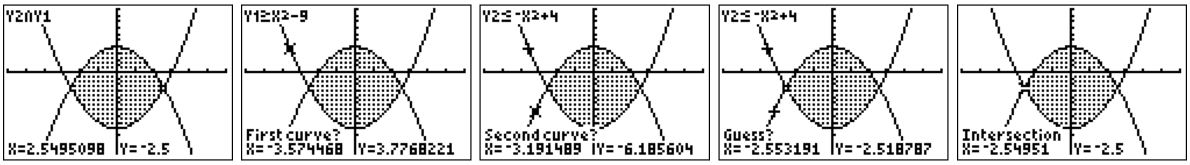
For linear inequalities it is possible to calculate the vertices the shaded area with **PoI-Trace (ALPHA F3)**: ◀ ▶ = change the second function & ▲ ▼ = change the first function.

You can store a selected vertex by pressing **STO** ▶. The coordinates of the vertex will automatically be stored in the lists **INEQX** and **INEQY**.



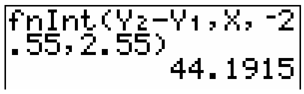
Example 2

Let's try to find the area between the functions $f(x) = x^2 - 9$ and $g(x) = -x^2 + 4$. For non linear functions it is not always possible to find the intersection points through Inequality Graphing. In such a case we need to use **5: intersect** of the graphical **CALC** menu.



To approximate the area we can use the **fnInt** command. The calculations above are also numerical approximations of the intersection points $x_1 = -\sqrt{\frac{13}{2}} \approx -2.55$ and $x_2 = \sqrt{\frac{13}{2}} \approx 2.55$.

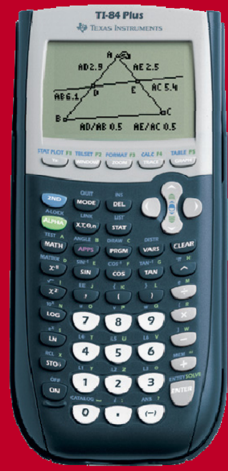
$\int_{-2.55}^{2.55} \left(g(x) - f(x)\right)dx$ is a good approximation of this area.



The intention of this booklet is offering an introduction to the use of the TI-84 Plus and its usability in the classroom.

The most important possibilities are discussed using mathematical examples without stressing the key press history too much.

In addition an example of linear programming is treated to show the various approaches for solving a problem with the TI-84 Plus, as well as the working of the applications Inequality Graphing and Transformation Graphing.



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